Treewidth of the Kneser Graph

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Contents

1 Abstract

In this paper, we present the results of the article "Treewidth of the Kneser Graph and the Erdős-Ko-Rado Theorem" by Harvey and Wood [5]. The treewidth of a graph is a measure of its complexity, and is difficult to compute in general. This work determines the treewidth of Kneser graphs, which are a common class of combinatorial graphs whose vertices are all the k element subsets of the numbers 1 to n, with edges only between subsets that are disjoint. We primarily focus our attention on proving their formula for the exact treewidth of a Kneser graph when n is large compared to k.

2 Introduction

2.1 Overview

Treewidth is a property of a graph that captures some sense of how "tree-like" it is. Halin[4] was the first to define treewidth, using a somewhat different notation from what we use today. The concept gained widespread attention for the role it played in Robertson and Seymour's influential papers on the Graph Minor Theorem[14]. Since that time, treewidth has found important applications in many diverse areas, including VLSI design (the Gate Matrix Layout problem)[1] and the Cholesky factorization of sparse symmetric matrices[1]. Linear time algorithms for some interesting problems are available when the graph family in question has bounded treewidth[1]. Our exploration here will center on the recent work of Harvey and Wood[5], wherein they find an exact expression for the treewidth of the Kneser graphs $\texttt{Kneser}(n, k)$ where $n \geq 4k^2 - 4k + 3$.

2.2 Tree Decompositions

We view a *graph* G as an ordered pair $G = (V, E)$, where V is a finite set of *vertices* and E is a set of 2-element subsets of V, called *edges*. Throughout this paper, our graphs will be finite, simple, and undirected. For more of the basic terminology regarding graphs, we refer the reader to the textbook by West[15]. A *tree decomposition* of a graph G is a triple (T, B, f) where T is a tree, B is a collection of subsets of $V(G)$ (these subsets are called *bags*), and $f : V(T) \to B$ is a surjective function that maps each vertex of T to a bag in B. Further, these properties must also hold:

1. the bags that contain a vertex $v \in V(G)$ induce a non-empty connected subtree of T. That is, for each $v \in V(G)$, let B_v be the set of all the bags containing v. Then the preimage $f^{-1}(B_v) = \{t \in V(T) : f(t) \in B_v\}$ is the vertex set of a non-empty connected subtree of T.

2. for each $vw \in E(G)$, some bag in B contains both v and w.

Figure 1 shows an example.

The *width* of a tree decomposition is the size of the largest bag, minus 1. In our example, as shown in Figure 2, the treewidth is 2. However, a tree decomposition need not be minimum-width - Figure 3 shows a different, valid, tree decomposition for the same graph that has a width of 3. The *treewidth* of a graph G, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of G. Determining the treewidth of a graph in general is an

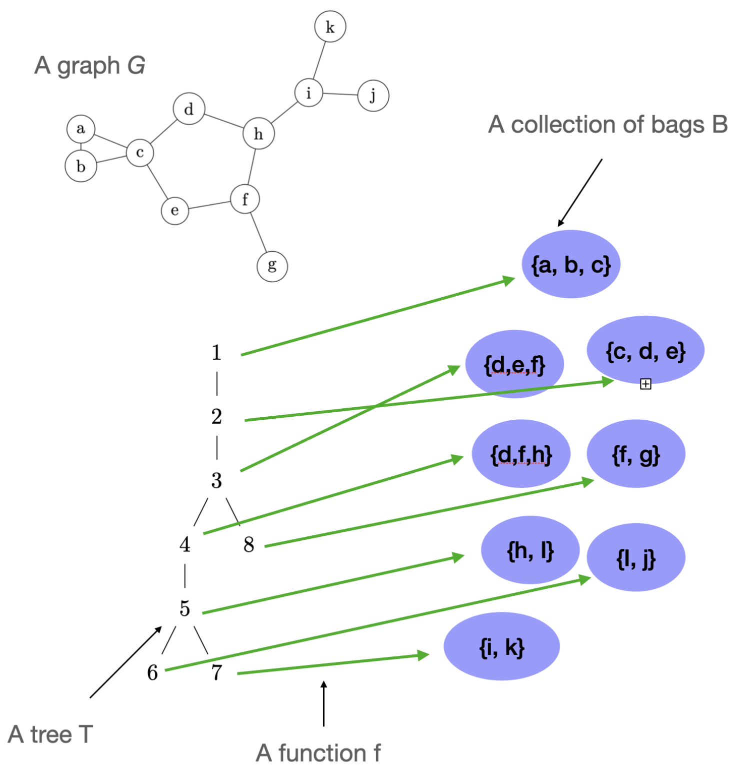

Figure 1: The various parts of a tree decomposition (T, B, f) for a graph G

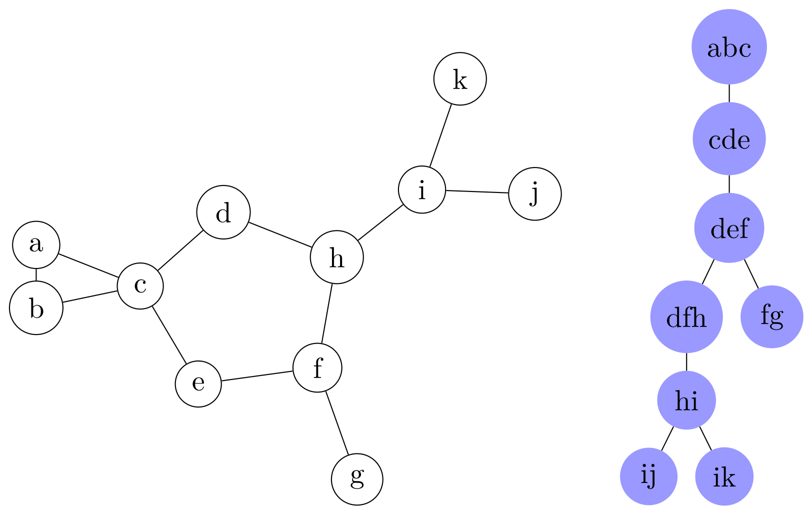

Figure 2: An example graph, with a minimum-width tree decomposition

NP-complete problem, though for certain families of graphs it is more easily computable.

2.3 Kneser Graphs

A *Kneser graph*, denoted $\texttt{Kneser}(n, k)$ is the graph with all the k-element subsets of $[n]$ as vertices, and edges only between sets that do not intersect. See Figure 3 for depictions of two small examples of Kneser graphs.

In the case of $k = 1$, $\texttt{Kneser}(n, k)$ is the complete graph K_n. At $k = 2$ the graph is an induced matching. We will usually assume herein that $n \geq 2k$.

2.4 Statement of Main Results

These are the theorems of Harvey and Wood that concern Kneser graphs:

Theorem 1. *[5, Theorem 1] Let G be a Kneser graph with $n \geq k^2 - 4k + 3$ and $k \geq 3$. Then*

$$tw(G) = \binom{n-1}{k} - 1$$

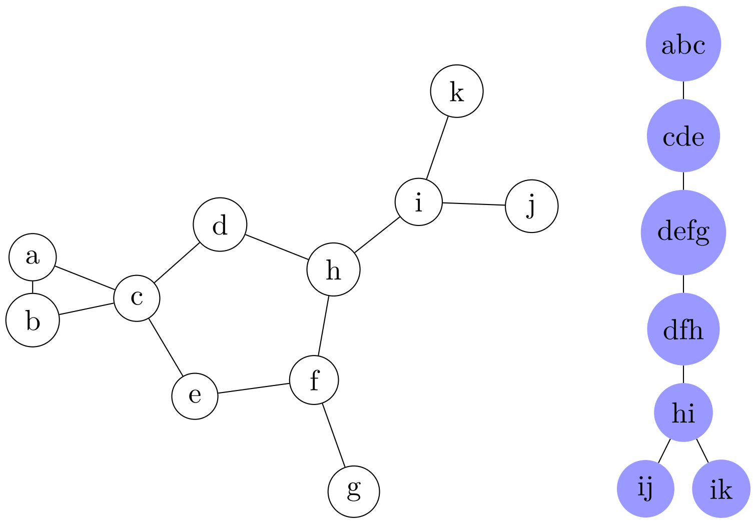

Figure 3: An example graph, with a valid, but non-minimum-width tree decomposition

Theorem 2. *[5, Theorem 2] Let G be a Kneser graph with $k = 2$. Then*

$$tw(G) = \begin{cases} 0 & \text{if} \quad n \leq 3 \\ 1 & \text{if} \quad n = 4 \\ 4 & \text{if} \quad n = 5 \\ \binom{n-1}{2} - 1 & \text{if} \quad n \geq 6 \end{cases}$$

We consider Theorem 1 to be the main result of interest, and so we will accordingly focus this project on presenting its proof.

3 Basic Properties of Treewidth

In this section we collect a number of basic facts about tree decompositions that will prove useful later.

3.1 Extreme cases and Non-Uniqueness

Lemma 1. *The maximum width tree decomposition of a graph with n nodes has width $n - 1$*

Proof. A tree with a single node that contains all vertices of the original graph is a valid tree decomposition, with width $n - 1$. No bag can contain

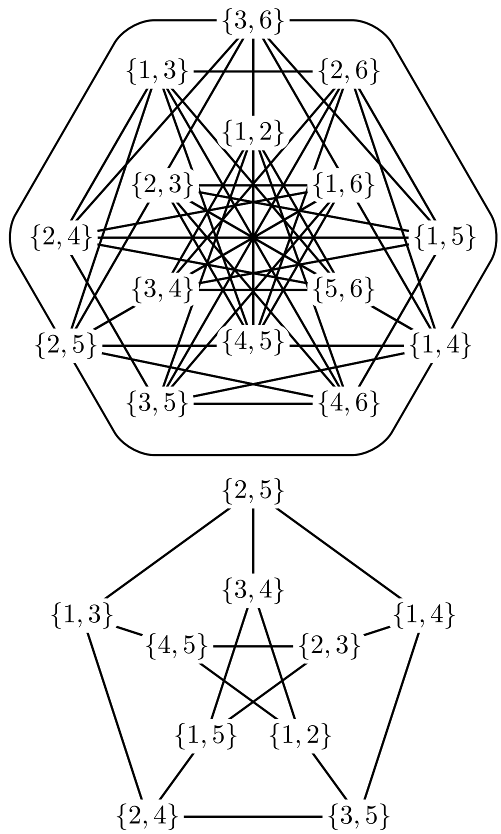

Figure 4: The graphs Kneser$(6, 2)$ and Kneser$(5, 2)$ (better known as the Petersen graph)

more than n vertices since there are only n distinct vertices in the graph, so there cannot be a tree with greater width. $\square$

Lemma 2. *A graph G with no edges has tree width 0.*

Proof. Suppose G is an empty graph with vertex set $V = \{v_1, ..., v_n\}$. Then (P_n, B, f) is a valid tree decomposition of width 0, where $P_n = ([n], \{\{i, j\} : |i - j| = 1\}$ is a path on n vertices, $B = \{\{v_i\} : i \in [n]\}$ is the set of all singletons of V, and the function f maps $i \mapsto \{v_i\}$ for all $i \in [n]$. We note that, since G has no edges, no bag needs more than one vertex. $\square$

Lemma 3. *The minimum-width tree decomposition of a graph is not necessarily unique.*

Proof. Consider the triangle K_3 with vertices {a, b, c}. A minimum-width tree for this graph has width 2 as it is a cycle. Note that both the graphs in Figure 5 are valid width 2 tree decompositions. □

Figure 5: Two minimum-width tree decompositions for K_3

3.2 Subgraphs and Components

Lemma 4. *The tree width of a subgraph is less than or equal to the tree width of the containing graph.*

Proof. Let G be a graph and T a minimal-width tree decomposition of G. Let $S \leq G$ be a subgraph of G. Then to form a tree decomposition of S, begin with T and delete any vertices of $G - S$ from the bags of T. The resulting tree T' has the same structure as T but its width is less than or equal to that of T.

□

Lemma 5. *The tree width of a disconnected graph G formed from components $G_1, G_2, ..., G_n$ is* $\max\{tw(G_1), tw(G_2), ..., tw(G_n)\}$.

Proof. Let $T_1, T_2, ..., T_n$ be the tree decompositions of the components of G. A tree decomposition for G can be formed by adding an edge from T_1 to each of the other $T_2, ..., T_n$. Since no new bags were added, nor new vertices added to existing bags, the tree width is the greatest width of any of $T_1, T_2, ..., T_n$. □

3.3 Paths and Cycles

In this section, we consider paths and cycles, and we determine their treewidth directly.

Lemma 6. *A path has tree width 1.*

Proof. Suppose P_n is a path with vertex set $V = \{v_1, ..., v_n\}$, where v_i, v_j are adjacent if and only if $|i-j| = 1$. Let $T = ([n-1], \{\{i,j\} : |i-j| = 1\}$ be a path on $n-1$ vertices, Then (T, B, f) is a valid tree decomposition of P_n, where $B = \{\{v_i, v_{i+1}\} : i \in [n-1]\}$, and the function f maps $i \mapsto \{v_i, v_{i+1}\}$

for all $i \in [n-1]$. This decomposition has width 1. We also note that there cannot be a decomposition with smaller width, since a single edge in a graph implies a tree width of at least 1, so P_n has tree width 1. $\square$

Determining the width of cycles is a bit more interesting. We will need to normalize our tree decompositions so that the bags all contain the same number of vertices, and so that these bags have small differences along edges. We begin with a definition.

A tree decomposition (T, B, f) of a graph G is said to be **normalized** if all the bags have the same cardinality and, for each edge $\{i, j\}$ in T, we have

$$|f(i) - f(j)| = |f(j) - f(i)| = 1.$$

It is interesting that we may normalize our tree decompositions without changing the width.

Lemma 7. *[6, Lemma 2.2] If a graph G has a tree decomposition of width k, then it also has a normalized tree decomposition of width k.*

Proof. Suppose G has a tree decomposition (T, B, f) of width k. If any bag has size $< k+1$, there must exist neighbors i, j somewhere in T where $|f(i)| = k+1 > |f(j)|$. So there exists some vertex $v \in f(i) - f(j)$. Adding v to the bag $f(j)$ increases $|f(j)|$ and still yields a valid decomposition. We may repeat until all bags have size $k+1$.

Now fix any edge $\{i, j\}$ in T. If $f(i) = f(j)$ then we can contract edge $\{i, j\}$ in T, labeling the new vertex i and obtaining a (new) tree T and a (still) valid decomposition. We may repeat until adjacent vertices in T have distinct bags.

Finally, fix any edge $\{i, j\}$ in T. If $|f(i) - f(j)| > 1$, there exists some vertex $u \in f(i) - f(j)$ and some vertex $v \in f(j) - f(i)$. Subdivide the edge $\{i, j\}$ and call the new vertex t, located between i and j in the (new) tree T. Associate t with the bag $f(t) = (f(i) - \{u\}) \cup \{v\}$. Now the (new) tree T has one new vertex and we still have a valid decomposition. Furthermore, $|f(i) - f(t)| = 1$ and $|f(t) - f(j)| = |f(i) - f(j)| - 1$. We may repeat until all edges have a difference of cardinality 1, as desired. $\square$

Lemma 8. *If a graph G has treewidth k, then the minimum degree of G is at most k.*

Proof. Suppose G has a tree decomposition (T, B, f) of width k. By the previous lemma, we may assume the decomposition is normalized. Let $\{i, j\}$ be any edge in T for which i is a leaf. Then the bag $f(i)$ contains a vertex

$v \notin f(j)$. So all the neighbors of v must be in $f(i)$ and therefore the degree of v can be at most k. □

Lemma 9. *A cycle has tree width 2.*

Proof. If not, there must exist a cycle of tree width 1. But the previous lemma would imply it has a vertex of degree at most 1, which is a contradiction. □

Lemma 10. *A connected graph with at least 2 vertices has tree width 1 if and only if it is a tree.*

Proof. ($\Leftarrow$): Let T be a tree with $|V(T)| \geq 2$. We proceed by induction.

Base case: Let T be a tree with 2 vertices, say u and v. Note that T has a valid tree decomposition with a single bag $\{u, v\}$, and this decomposition has width 1.

Inductive case: Suppose $G_1, ..., G_n$ are trees with $\text{tw}(G_x) = 1$ for all x in $[n]$, and let T_x be the respective minimal tree decomposition for each G_x. Let v be a new vertex, connected to each of $G_1, ..., G_n$ by new edges $vg_1, ..., vg_n$. Let this new graph be G. We construct a new tree decomposition T for G as follows: add a new node $\{v\}$, and for each edge $vg_1, vg_2, ..., vg_n$ in G, add nodes $\{v, g_1\}, \{v, g_2\}, ..., \{v, g_n\}$, each with an edge to $\{v\}$, and with an edge to its respective vertex in $T_1, T_2, ..., T_n$. None of the new bags we added had more than two elements, so the resulting tree has width 1.

($\Rightarrow$): Suppose G has tree width 1, and suppose by way of contradiction that G is not a tree. Then G contains a cycle and so its tree width is at least 2. This is a contradiction. □

3.4 Complete graphs and Cliques

Lemma 11. *A complete graph K_n has tree width $n - 1$.*

Proof. We proceed by induction.

Base case: K_2 has tree width 1.

Inductive case: Suppose K_{n-1} has tree width $n - 2$. Let T_{n-1} be a minimal tree decomposition for K_{n-2}. If we add a vertex v to K_{n-1} and connect it with all existing vertices, those new edges must be reflected in the tree decomposition. Since there is an edge to every vertex, for every $x \in V(K_{n-1})$, we must have xv in some bag. To avoid increasing the tree width, there must be at least two bags in the tree decomposition for K_{n-1} that have cardinality less than $n-1$, which together contain all the vertices

- by adding v to them, we will satisfy rule 2. Since we add v to at least two of these, those two must be connected. However, each of these is connected to a bag of size $n-1$, by rule 1. So we have created a cycle. So we must instead add v to a bag of size $n-1$, which satisfies the rules and produces a tree width of $n-1$. □

Lemma 12. *Any graph that contains a clique of size k has tree width at least $k-1$.*

Proof. If a graph G has a clique of size k, the clique has treewidth $k-1$. The clique is a subgraph of G, so $\text{tw}(G) \geq k-1$. □

4 Separators and Shadows

From this point, all graphs are Kneser(n, k) graphs, and we use G, n, and k implicitly. Let $\Delta(G)$ be the maximum degree, $\delta(G)$ the minimum degree, and $\alpha(G)$ the cardinality of the largest independent set. (Recall a set of vertices is *independent* if the induced subgraph contains no edges.)

4.1 Separators

A *p-separator of order k* for any $p \in [\frac{2}{3}, 1)$, is a set $X \subset V(G)$ with $|X| \leq k$ such that no component of $G - X$ contains more than $p|G - X|$ vertices. See Figure 6 for an example.

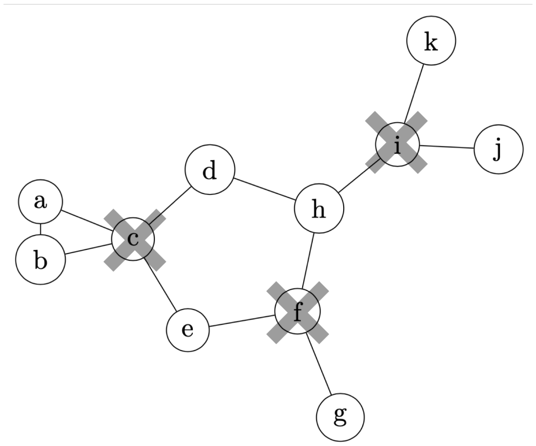

Figure 6: Note $X = \{c, f, i\}$ is a $\frac{2}{3}$-separator of order 3, since no component of $G - X$ has more than $\frac{2}{3}|G - X| = 16/3$ vertices.

Theorem 3. *[5, Theorem 4] (c.f. [14]) For each* $p \in [\frac{2}{3}, 1)$*, every graph* G *has a* p*-separator of order* $tw(G) + 1$*.*

Lemma 13. *[5, Lemma 5] Let* X *be a* p*-separator. Then* $V(G - X)$ *can be partitioned into two parts* A *and* B*, with no edge between* A *and* B *such that*

- $(1 - p)|G - X| \leq |A| \leq \frac{1}{2}|G - X|$
- $\frac{1}{2}|G - X| \leq |B| \leq p|G - X|$.

In other words, we have:

$$(1 - p)|G - X| \leq |A| \leq \frac{1}{2}|G - X| \leq |B| \leq p|G - X|.$$

Proof. Since X is a p-separator, the vertices of $G - X$ are partitioned into components $V_1, \ldots, V_c$ such that $c \geq 2$ and, for any $1 \leq i < j \leq c$, the following two conditions hold:

1. there are no edges between V_i and V_j, and
2. $1 \leq |V_i| \leq |V_j| \leq p|G - X|$.

If $c = 2$, our partition has just two parts. In this case, we can let $A = V_2$ and $B = V_1$ and the result follows immediately.

If $c \geq 3$, the pigeonhole principle implies $|V_{c-1} \cup V_c| \leq \frac{2}{3}|G - X|$. (Otherwise, $|V_{c-1} \cup V_c| > \frac{2}{3}|G - X|$, so one of $|V_{c-1}|$ and $|V_c|$ exceeds $\frac{1}{3}|G - x|$. But then $V_1 > \frac{1}{3}|G - x|$, and

$$|G - X| \geq |V_1| + |V_{c-1} \cup V_c| > (\tfrac{1}{3} + \tfrac{2}{3})|G - X|,$$

which is impossible.) So we can combine the 2 smallest components V_{c-1} and V_c (and reorder if necessary) to obtain a new partition with $c - 1$ sets satisfying the two conditions above. Repeat until we obtain a partition with 2 parts, as desired. □

Theorem 4. *(Erdős-Ko-Rado)[5, Theorem 6] (c.f. [8][2]). Let* G *be* $\texttt{Kneser}(n, k)$ *for some* $n \geq 2k$*. Then* $\alpha(G) = \binom{n-1}{k-1}$*. If* $n \geq 2k + 1$ *and* $\mathcal{A}$ *is an independent set such that* $|\mathcal{A}| = \binom{n-1}{k-1}$*, then* $\mathcal{A} = \{v | i \in v\}$ *for a fixed element* $i \in [n]$*.*

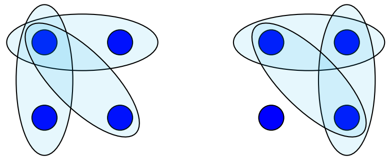

Figure 7: Two examples of intersecting families of two-element subsets of a four-element set

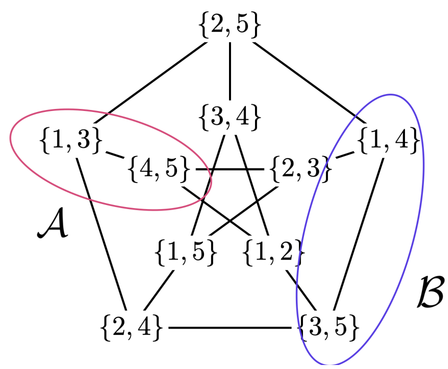

Figure 8: Cross-intersecting families $\mathcal{A}$ and $\mathcal{B}$

Harvey and Wood use this definition, though the original Erdős-Ko-Rado theorem defines $\mathcal{A}$ equivalently based on k-sets in $[n]$. See Figure 7 for an example of intersecting families.

In a Kneser graph G, let $\mathcal{A}, \mathcal{B}$ be sets of vertices of G. Then $\mathcal{A}$ and $\mathcal{B}$ are *cross-intersecting families* if there are no edges between the vertices in $\mathcal{A}$ and the vertices in $\mathcal{B}$. See Figure 8 for an example.

Theorem 5. *(Erdős-Ko-Rado for Cross-Intersecting Families)[5, Theorem 7] (c.f. [11][13]) Let $n \geq 2k$ and let $\mathcal{A}$ and $\mathcal{B}$ be cross-intersecting families in the Kneser graph G. Then $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$. If $n \leq 2k+1$ and $\mathcal{A}$ and $\mathcal{B}$ are cross-intersecting families such that $|\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}^2$, then $\mathcal{A} = \mathcal{B} = \{v | i \in v\}$ for a fixed element $i \in [n]$.*

This forumulation, specific to Kneser graphs, is also due to Harvey and Wood, and is less general than the original theorem by Pyber[13].

Let X be a $\frac{2}{3}$-separator that partitions $G - X$ into A and B as in Lemma 13. So there are no edges between A and B, so they are cross-intersecting families. We have that $|A| = c|G - X|$ where $\frac{1}{3} \leq c \leq \frac{1}{2}$. So by

Theorem 5, $c(1-c)|G-X|^2 \leq \binom{n-1}{k-1}^2$. Since $\frac{1}{3} \leq c \leq \frac{1}{2}$,

$$|G-X|^2 \leq \frac{\binom{k-1}{n-1}^2}{c(1-c)} \leq \frac{\binom{n-1}{k-1}^2}{\frac{1}{3}(1-\frac{1}{3})}$$
$$= \frac{\binom{n-1}{k-1}^2}{\frac{2}{9}} = 9\frac{\binom{n-1}{k-1}^2}{2}.$$

So,

$$|G-X| \leq 3\frac{\binom{n-1}{k-1}}{\sqrt{2}}.$$

Therefore, $|G-X| \leq \frac{3}{\sqrt{2}}\binom{n-1}{k-1}$, which provides a lower bound on $|X|$, which (by Theorem 3) is also a lower bound on the treewidth. Therefore $\mathrm{tw}(G) \geq \binom{n}{k} - \frac{3}{\sqrt{2}}\binom{n-1}{k-1} - 1$.

Furthermore, A and B partition $V(G-X)$ and so are disjoint, though that is not a requirement of Theorem 5. Theorem 5 also shows that when $|\mathcal{A}||\mathcal{B}|$ is maximized, $\mathcal{A} = \mathcal{B}$. Because of this, we can improve on the naive bound on tw(G) above.

We need a few more definitions before the next lemmas.

4.2 Colex ordering

We refer to sets of size k with elements drawn from $[n]$ as *k-sets in $[n]$*, or simply as k-sets when the source set is clear from context.

The *colexicographic ("colex") ordering* of k-sets of $[n]$ is a strict total order in which a k-set x is less than the k-set y when $max(x-y) < \max(y-x)$.

A set X of k-sets is *first* if X consists of the first $|X|$ sets in the colex ordering of all k-sets in $[n]$. Note that in the colex ordering of k-sets in $[n]$ (Figure 9 shows an example), if we consider the k-sets in $[i]$ where $i < n$, we see that they all come before any k-set containing an element greater than i. This is because if x, y are k-sets in $[i], [n]$ respectively, and $j \in y > i$, then $\max(x-y) \leq \max(x) \leq i$, and $\max(y-x) \geq$ since $j \in y-x$. This will be useful in reasoning about first sets in the proof of the lower bound in Theorem 1.

4.3 Shadows

Let X be a set of k-sets in $[n]$. For any $c \leq k$, the *c-shadow* of X is all the c-sets that are each a subset of some set in X. That is, the c-shadow of X

Standard sort	Colex sort
$\{1,2,3\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,2,4\}$
$\{1,2,5\}$	$\{1,3,4\}$
$\{1,2,6\}$	$\{2,3,4\}$
$\{1,3,4\}$	$\{1,2,5\}$
$\{1,3,5\}$	$\{1,3,5\}$
$\{1,3,6\}$	$\{2,3,5\}$
$\{1,4,5\}$	$\{1,4,5\}$
$\{1,4,6\}$	$\{2,4,5\}$
$\{1,5,6\}$	$\{3,4,5\}$
$\{2,3,4\}$	$\{1,2,6\}$
$\{2,3,5\}$	$\{1,3,6\}$
$\{2,3,6\}$	$\{2,3,6\}$
$\{2,4,5\}$	$\{1,4,6\}$
$\{2,4,6\}$	$\{2,4,6\}$
$\{2,5,6\}$	$\{3,4,6\}$
$\{3,4,5\}$	$\{1,5,6\}$
$\{3,4,6\}$	$\{2,5,6\}$
$\{3,5,6\}$	$\{3,5,6\}$
$\{4,5,6\}$	$\{4,5,6\}$

Figure 9: Standard and colexecographic ("colex") ordering of 3-sets in [6]

Set	2-set
$\{2,3,6\}$	$\{2,3\}$
	$\{2,6\}$
	$\{3,6\}$
$\{2,4,6\}$	$\{2,4\}$
	$\{2,6\}$
	$\{4,6\}$
$\{2,3,4\}$	$\{2,3\}$
	$\{2,4\}$
	$\{3,4\}$

Figure 10: An example of the 2-shadow of the set $\{\{2,3,6\},\{2,4,6\},\{2,3,4\}\}$. The complete shadow in colex order is: $\{\{2,3\},\{2,4\},\{3,4\},\{2,6\},\{3,6\},\{4,6\}\}$

is $\{s : |s| = c, \exists x \in X. c \subseteq x\}$. For example, see Figure 10.

Lemma 14. *(A first set minimizes the shadow) [5, Lemma 8], (c.f. [9], [7], [3]). Let X be a set of k-sets in $[n]$, $c \leq k$, and S the c-shadow of X. Suppose $|X|$ is fixed but X is not. Then $|S|$ is minimized when X is first.*

4.4 Complements of k-sets

For x a k-set in $[n]$, the *complement* of x is the $(n-k)$-set $([n] - x)$, and the *complement* of a set of k-sets X (written $\overline{X}$) is $\{y : \exists x \in X. y = \overline{x}\}$. For example, if $x = \{1, 3\}$ is a 2-set in $[6]$, $\overline{x} = \{2, 4, 5, 6\}$.

5 Upper bound for treewidth

We begin with a lemma that gives us an upper bound for any graph. We will then use that, along with properties of Kneser graphs, to prove a tighter bound for Kneser graphs.

Lemma 15. *[5, Lemma 9]. If H is a graph, $tw(H) \leq \max\{\Delta(H), |V(H)| - \alpha(H) - 1\}$.*

Proof. We can construct a tree decomposition for any graph H by taking a maximum independent set I of size $\alpha = \alpha(H)$, and creating $\alpha + 1$ bags: first, a bag that contains all of $H - I$, and then α bags I_i for $i \in [\alpha]$, which partition I. To each I_i we add the neighborhood of the i^{th} vertex in I. We create a tree that is a star, with central node mapping to the $H - I$ bag, and then a leaf for each of the 1-element bags. See Figure 11 for an example. Any induced subgraph that contains the center is necessarily connected, and any edge vw has vertices in at most one leaf bag, since the leaf bags were all formed from an independent set. So this is a valid tree decomposition. The size of the center bag is $|V(H)| - \alpha$, and since the leaf bags are formed from a single vertex and its neighborhood, they are of size at most $\Delta(H)$. $\square$

Lemma 16. *[5, Lemma 10]. If G is a Kneser graph with $k \geq 2$ and $n \geq 2k+1$, then $tw(G) \leq \binom{n}{k-1} - 1$.*

Proof. Since $n \geq 2k + 1$, we have by Lemma 15 and Theorem 4,

$$\begin{aligned}\text{tw}(G) &\leq \max\{\Delta(G), |V(G)| - \alpha(G) - 1\} \\ &= \max\left\{\binom{n-k}{k}, \binom{n}{k} - \binom{n-1}{k-1} - 1\right\}.\end{aligned}$$

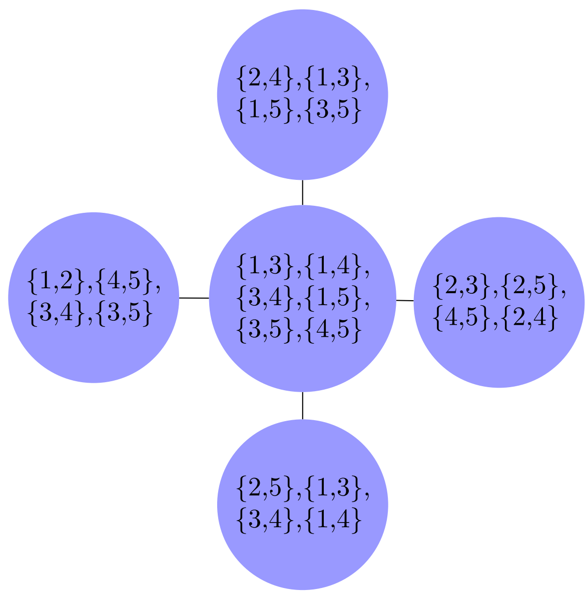

Figure 11: A sample star construction for the upper bound of Lemma 15. The source graph is the Petersen graph, Kneser(5,2), which is well known to have treewidth 5

Since $k \geq 2$,

$$\binom{n-k}{k} \leq \binom{n-2}{k} < \binom{n-1}{k}$$

and so

$$\binom{n-k}{k} \leq \binom{n-1}{k} - 1.$$

Therefore, $\text{tw}(G) \leq \binom{n-1}{k} - 1$. □

6 Lower bound for treewidth

We now turn our attention to the lower bound for Theorem 1.

Lemma 17. *([5, Lemma 11]). Let X be a p-separator of the Kneser graph G. If $n \geq max(4k^2 - 4k + 3, \frac{1}{1-p}(k^2-1)+2)$, then $|X| \geq \binom{n-1}{k})$.*

Proof. Suppose, by way of contradiction, $|X| < \binom{n-1}{k}$. Then $|G - X| > \binom{n-1}{k-1}$. Lemma 13 tells us $G - X$ can be partitioned into two parts A and B such that $(1-p)|G-X| \leq |A| \leq \frac{1}{2}|G-X| \leq |B| \leq p|G-X|$ and there are no edges between A and B. For any $i \in [n]$, let $A_i := \{v \in A : i \in v\}$, and define $A_{-i} := \{v \in A : i \notin v\}$. Note that A_i and A_{-i} partition the set A for any i. Define similar sets for B.

Claim 1. There exists some i such that $|B_i| \geq \frac{1}{k}|B|$.

Proof. Because $|A| \geq (1-p)|G-X| > 0$, there A contains at least one vertex v. Without loss of generality, $v = \{1, ..., k\}$. No $w \in B$ is adjacent to v, so w and v intersect. So every w contains at least one of $1, ..., k$. Therefore at least one of these elements appears in at least $\frac{1}{k}|B|$ of the vertices of B. □

Without loss of generality, $|B_n| \geq \frac{1}{k}|B|$.

Claim 2. $|B_n| > \binom{n-3}{k-2} + \binom{n-1}{k-2}$.

Proof. We know $|B| \geq \frac{1}{2}|G-X| \geq \frac{1}{2}\binom{n-1}{k-1}$. So by Claim 1 and our subsequent assumption, $|B_n| \geq \frac{1}{k}|B| \geq \frac{1}{2k}|G-X| \geq \frac{1}{2k}\binom{n-1}{k-1}$. Suppose, by way of contradiction, that $|B_n| \leq \binom{n-3}{k-2} + \binom{n-2}{k-2}$. So

$$\frac{1}{2k}\binom{n-1}{k-1} \leq \binom{n-3}{k-2} + \binom{n-2}{k-2}.$$

So

$$(n-1)! \leq 2k(k-1)((n-k)(n-3)! + (n-2)!).$$

Therefore

$$n^2-3n+2 = (n-1)(n-2) \leq 2k(k-1)(2n-k-2) = 4k^2n-4kn-2k^3-2k^2+4k.$$

So $n^2+(4k-4k^2-3)n+2k^3+2k^2-4k+2 \leq 0$. Because $n \geq 4k^2-4k+3$, we have that $2k^3+2k^2-4k+2 \leq 0$. But $k \geq 1$, so we have a contradiction. □

We examine the set of complements of vertices in A that do not contain n, $\overline{A_{-n}}$. Note that n is contained in each set in $\overline{A_{-n}}$. Define $\overline{A_{-n}}^* := \{\overline{v}-n : \overline{v} \in \overline{A_{-n}}\}$. That is, remove n from each set in $\overline{A_{-n}}$, and note there is a one-to-one correspondence between $(n-k)$-sets in $\overline{A_{-n}}$ and $(n-k-1)$-sets in $\overline{A_{-n}}^*$. Similarly, define $B_n^* := \{v - n : v \in B_n\}$. That is, delete n from each vertex of B_n. These new sets are $(k-1)$-sets in $[n-1]$.

Claim 3. If $v^* \in B_n^*$ and $\overline{w}^* \in \overline{A_{-n}}^*$, then $v^* \not\subseteq \overline{w}^*$.

Proof. Suppose, by way of contradiction, that $v^* \subseteq \overline{w}^*$. Then $v \subseteq \overline{w}$ by restoring n to both sets. So v and w are adjacent. But $v \in B_n \subseteq B$ and $w \in A_n \subseteq A$, which is a contradiction.

Let S be the $(k-1)$-shadow of $\overline{A_{-n}}^*$. So $v \in B_n^*$ implies $v \notin S$, by Claim 3. Therefore,

$$B_n^* \subseteq \binom{[n-1]}{k-1} - S.$$

Thus if we take $|S|$ to be minimized, we have an upper bound for $|B_n^*|$, and by Lemma 14, $|S|$ is minimized when $\overline{A_{-n}}^*$ is first. □

Claim 4. $|A_{-n}| \leq \binom{n-3}{k-2}$.

Proof. We know $|A_{-n}| = |\overline{A_{-n}}| = |\overline{A_{-n}}^*|$, so we need only show that $|\overline{A_{-n}}^*| \leq \binom{n-3}{k-2}$. Suppose, by way of contradiction, that $|\overline{A_{-n}}^*| \geq \binom{n-3}{k-2} = \binom{n-3}{n-k-1}$. Firstly, we show that $|S| \geq \binom{n-3}{k-1}$. Since we are proving a lower bound, we can take $|S|$ to be minimized, and so we can assume that $\overline{A_{-n}}^*$ is first, and contains the first $\binom{n-3}{n-k-1}$-sets on $[n-3]$, because there are $\binom{n-3}{n-k-1}$ such sets, which come before all other sets in the ordering. Therefore all $(k-1)$-sets in $[n-3]$ are in S, and so $|S| \geq \binom{n-3}{k-1}$. Therefore $|B_n^*| \leq \binom{n-1}{k-1} - \binom{n-3}{k-1} = \binom{n-3}{k-2} + \binom{n-2}{k-2}$. But Claim 2 tells us $|B_n^*| = |B_n| > \binom{n-3}{k-2} + \binom{n-2}{k-2}$, which is a contradiction. □

Claim 5. $|A_n| \geq \frac{k}{k+1}|A|$.

Proof. We begin by showing that $|A_n| \geq k|A_{-n}|$. Suppose not. Then by Claim 4, $|A| = |A_n| + |A_{-n}| < (k+1)|A_{-n}| \leq (k+1)\binom{n-3}{k-2}$. But $|A| \geq (1-p)|G-X|$. So $(1-p)\binom{n-1}{k-1} < (k+1)\binom{n-3}{k-2}$. Then $(n-1)(n-2) < \frac{1}{1-p}(k+1)(k-1)(n-k) \leq \frac{1}{1-p}(k+1)(k-1)(n-2)$. So we have $n < \frac{1}{1-p}(k^2-1)+1$, which violates the lower bound on n. So $|A_n| \geq k|A_{-n}| = k(|A|-|A_n|)$, and $(k+1)|A_n| \geq k|A|$ as required. □

Claim 6. $B_n = B$.

Proof. Suppose not. Then there is some vertex $v \in B$ such that $n \notin v$. By definition, each $w \in A_n$ contains n, and some other element of v, because vw is not an edge. We can construct any vertex of A_n thus: choose element n, then pick one of the k elements of v, and choose the remaining $k-2$ elements from the remaining $n-2$ elements of $[n]$. So

$$|A_n| \leq 1 \cdot k\binom{n-2}{k-2}.$$

We have overcounted some vertices of A_n, so this is a a weak upper bound. We know $|A| \geq (1-p)|G-X| \geq (1-p)\binom{n-1}{k-1}$. So by Claim 5,

$$\frac{(1-p)k}{(k+1)}\binom{n-1}{k-1} \leq \frac{k}{k+1}|A| \leq k\binom{n-2}{k-2}.$$

Therefore $\frac{n-1}{k-1} \leq \frac{1}{1-p}(k+1)$ and $n \leq \frac{1}{1-p}(k^2-1)+1$, which contradicts our lower bound on n. □

Claim 7. $A_n = A$.

Proof. We proceed along the same lines as Claim 6. Suppose our claim does not hold and we have $v \in A$ such that $n \notin v$. By Claim 6, $|B_n| = |B| \geq \frac{1}{2}\binom{n-1}{k-1}$, and we have an upper bound on $|B_n|$ equal to the upper bound on $|A_n|$ in the previous proof. Therefore

$$\frac{1}{2}\binom{n-1}{k-1} \leq |B| = |B_n| \leq k\binom{n-2}{k-2},$$

and so $n \leq 2k(k-1)+1$, which contradicts the lower bound on n. □

By Claims 6 and 7, we have that every vertex in $G - X = A \cup B$ contains n. Therefore $|G - X| \leq \binom{n-1}{k-1}$ and $|X| \geq \binom{n-1}{k}$, and we have a contradiction. By Lemma 17, if X is a $\frac{2}{3}$-separator of the Kneser graph G and $n \geq 4k^2 - 4k + 3$, $|X| \geq \binom{n-1}{k}$. So by Theorem 3, $\mathrm{tw}(G) \geq \binom{n-1}{k} - 1$, which completes the proof of Theorem 1. □

7 Conclusions and directions

We have defined and shown examples of treewidth in graphs, and we have proved, in the fashion of Harvey and Wood, the exact treewidth for the Kneser graph where $n \geq 4k^2 - 4k + 3$. Harvey and Wood conjecture[5, pg. 9] that this same formula also holds for $n \geq 3k$ and $k \geq 2$, and possibly for $n \geq 3k - 1$ with the Petersen graph as the only exception.

Other authors (e.g. [12][10]) have extended the work of Harvey and Wood to *generalized* Kneser graphs. A generalized Kneser graph is one that adds a third parameter t, where the vertices may be joined if and only if the size of their intersection is less than t - a `Kneser`(n,k) graph is then a $K(n, k, 1)$ generalized Kneser graph.

Another related family of graphs is the family of generalized Johnson graphs. Here, $J(n, k, i)$ denotes the graph whose vertices are the k-element subsets of an n-element set, and where two vertices are adjacent whenever the intersection of their subsets has cardinality *exactly* i. It would be interesting to know if the techniques here could provide insight into the treewidth of generalized Johnson graphs.

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